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SYMMETRY AND CHAOS

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Abstract—The dynamical model of a particle moving in the magnetic field and wave packet field is discussed. Particle trajectories produce tilings with the same quasisymmetry on the phase plane as is, for example, in quasicrystals. This makes it possible to find the dynamical generator of quasisymmetry and to consider the quasisymmetry onset as the result of a weak interaction between a rotational symmetry and a translational symmetry. Such an interaction produces thin channels of chaotic dynamics of the particle in the phase space (stochastic web) which realizes the tiling of the plane with quasisymmetry.

1. INTRODUCTION

Attempts to look into the secrets of the geometry of the world in which we live began far back in time. There are good reasons why the greatest discoveries that drastically changed our concepts of the laws of nature have become milestones along this road. Much work is still being done in the same direction, since it is believed that better understanding of the properties of fields and particles should explain typical features of the micro and macroworld. This is, however, but one side of the medal. Its other side is art, where no smaller efforts were taken to understand the geometric harmony of the world, to unravel the mysteries of patterns, configurations, structures the nature itself creates. These two trends of search have never moved far apart, and from time to time there appeared personalities who managed to make discoveries in both geometry and art. Long is the list from unknown artists of ancient ages to Plato, Archimedes, Leonardo da Vinci, Kepler and Dürer—of those who had the same goal and strove to understand symmetry. Most intricate oriental ornaments are the best examples of works of art to emphasize which certain symmetry laws should be known.

Symmetry of patterns, symmetry of tilings, theory of tilings and dense packings on the plane and in space are only a few related branches in the studies of nature; joined with them are other branches known better (symmetry of crystals, symmetry of structures in fluids) or worse (symmetry of foam, symmetry of clouds). Unity of symmetry laws continuously makes this list longer and longer, adding to it purely applied problems [1, 2] as well.

Simpler laws of symmetry are associated with reflections of figures, with their rotations and translations; for instance, a two-dimensional (plane) ornament invariant to arbitrary translations may have 17 various types (groups) of symmetry. All of them were known by ancient Egyptian craftsmen. Eleven groups are found in the Moorish ornaments of Alhambra (Granada, Spain), 5 groups, additional to these 11, exist in African handicraft articles in Southern Sakhara, and one more, the last group, was found among the Chinese ornaments. All these groups of symmetry are presented in the work of Dutch artist M. C. Escher (1898–1972). Many of these symmetries were independently discovered by him.

It is easy to create tilings which remain symmetric when turned through the angle $\alpha = 2\pi/q$, where q is an integer (q-fold rotation symmetry). However, it is much more difficult to combine both types of symmetry in tilings: translational and rotational. This can only be done for $q = 2$, 3, 4, 6. That is why it was commonly accepted up to 1984 that crystals of the 5-fold symmetry do not exist. In 1984 Shechtman and his group produced—by cooling the alloy of Al and Mn—a new substance with the 5-fold symmetry as the X-ray spectroscopy pattern showed [3]. This alloy, also called Shechtmanite, is somewhat intermediate between crystals and liquids. This was not only a revolutionary step in crystallography, but also gave a new impetus to the studies of symmetry properties in nature.

The plane cannot be densely tiled with regular pentagons only, without gaps, however, the pentagons can be used as the basic elements, and it was Kepler and Dürer who proved it. After Penrose's [4], pentahedric tilings have been thoroughly analyzed by many professionals and amateurs [5, 6]. A world of new possibilities has opened up—the world of nonperiodic tilings with highly ordered-symmetry motifs. What is the origin of new types of symmetry which we call a symmetry of quasicrystal type and could they be associated with real-world physical processes?

SECTION 2

A possible answer to this question was formulated when quite a different problem was being considered at the Space Research Institute, U.S.S.R. Academy of Sciences. We know that the very same symmetry laws may manifest themselves in the phenomena of having nothing in common, at first glance. In 1959 one of the authors (R.Z.S.) described the model of collisionless shock waves and suggested a mechanism by which particles are accelerated before the front of the shock moving perpendicularly to the magnetic field [7] (now this mechanism is known as "surfatron"). If the magnetic field is constant and uniform then, in the plane perpendicular to the magnetic field, particles are moving along circular orbits. Imagine that near a particle there forms a wall parallel to the magnetic field. Then, the particle will be regularly reflected from the wall and drift along it perpendicularly to the magnetic field (Fig. 1), and its orbits will be arcs of a circle. If the wall is moving perpendicularly to the magnetic field the particle gains energy in each collision. The shock front acts as a wall. The magnetic field is a factor ensuring multiple collisions since it is due to the fact that after each collision the particle returns to the shock front. The moving shock wave affects the particle even if it is an untrapped particle, that is, when its energy exceeds the energy potential of the wave. Each time a particle passes above the crest of the wave it loses energy, moving in the same direction as the wave, or gains energy meeting the wave. The balance of these two effects depends on the relationship between the phases of a particle and a wave. If instead of one wave we consider a superposition of many plane waves (wave packet) there forms a very intricate dynamic picture which has been an object of study for many years in the context of various problems in plasma physics and astrophysics. It is one of these cases that is related with the problem of symmetry of plane tilings [8–10]. The wave packet may be arranged so that the particle undergoes very short-lived collisions (kicks) following equal time intervals. If, during the period of particle rotation in the magnetic field, exactly q kicks occur, the resonance of order q is said to take place. The number q, as we shall see below, will be fundamental for describing the dynamics of particles.

SECTION 3

When a particle is moving in complex fields its trajectory could be very intricate, though the "intricacy" may only seem to exist and may result from the combination of several simple motions. Since the trajectory of the particle is uniquely determined by setting the initial coordinates and impulses, it is usually represented in the phase space of all coordinates and impulses of the particle. However, now it is difficult to present the trajectory graphically, that is, the phase space is six-dimensional. Instead, Poincaré mappings are used. An oriented surface or plane is chosen and points of puncture of the plane by the trajectory in a certain direction are marked on it (Fig. 2). Thus, a set of punctured points forms. It is the function with which the position of a point the trajectory punctures on the plane can be estimated from the position of the previous one that specifies the Poincaré mapping:

$$\mathbf{R}_{n+1} = \hat{M}\mathbf{R}_n, \tag{1}$$

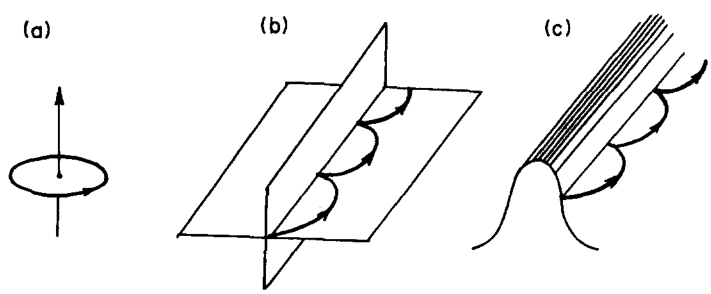

Fig. 1. Particle drift in a magnetic field: (a) free particle rotation; (b) particle drift along the reflecting wall; (c) particle drift along a wave front.

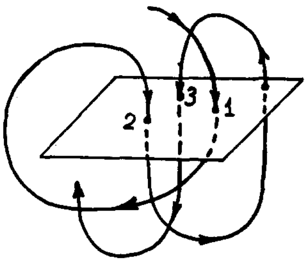

Fig. 2. Poincaré mapping; 1, 2, 3—consequent points of the mapping.

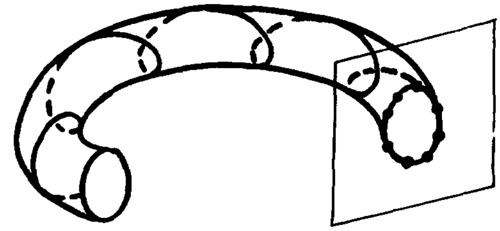

Fig. 3. Invariant curve formation in the plane of torus section.

where $\mathbf{R}_n$ is the vector that determines the position of the nth punctured point on the chosen plane; $\hat{M}$ is the matrix connecting two consecutive punctured points.

A set of punctured points produces an image of the mapping. If all the points of the image, for instance, are located along some closed curve, it means that the trajectory is winding on an invariant torus, whereas the curve itself is also invariant and results from the torus intersection with the plane chosen (Fig. 3). Here the word "invariant" is very important, meaning that the position of the torus and the curve is not time dependent. Different trajectories result for different initial conditions. These trajectories generate different invariant curves on the plane intersecting the tori. All this leads to a certain pattern of invariant curves on the chosen phase plane, which builds a phase portrait of the particle. It is this phase portrait that will help us judge the basic dynamic properties of particles or more complicated systems. For instance, the phase plane of the pendulum is made of closed curves, which represent its oscillations and open curves, which represent its rotations [Fig. 4(a)]. Two different types of motion—oscillations and rotations—are separated by a single curve, a separatrix which belongs to neither.

Extremely essential is the role of the separatrix in the modern theory of dynamic systems. The problem is that the vicinity of the separatrix is unusually sensitive to perturbations, however small they are. Indeed, it is here that a slight "stirring" of the initial condition may replace one type of motion with another. Hence if the pendulum is subjected to some arbitrary perturbation, periodic in time, then for the initial conditions near the separatrix, the pendulum trajectory will appear very unusual. Depending on the difference between the pendulum phase and the external force phase, the pendulum will be either oscillating or rotating. Even the very number of oscillations or rotations—between two changes of the mode—will vary without any order. As the analysis shows, such motion of the pendulum is chaotic as in no way does it differ from a certain random process [11]. Though a regular periodic force acts upon the pendulum, the latter behaves, however, as if this force is random. As a result, the phase portrait of the pendulum subjected to some perturbation has one more type of trajectories—stochastic trajectories.

There is a range of nonzero measures near the separatrix, such that if the initial condition is within this range, the respective trajectory will be stochastic. This region is called a stochastic layer [Fig. 4(b)]. The smaller the perturbation, the thinner the layer. It is a very nontrivial fact that the onset of chaos may occur in a system without any random forces. The studies of several recent decades have demonstrated that there exists no surpassable boundary between the random and the nonrandom. The same physical system may behave regularly (for instance, its coordinates will vary quasiperiodically in time) or randomly (if some of its parameters will be changed a little).

The above described phenomenon of a very complex irregular motion arising in dynamic systems is now usually called chaos. The phenomenon is most widely applied to all branches of modern physics [11]. In the so-called Hamiltonian systems, chaos originates in stochastic layers near the separatrices, destroyed due to perturbations. That is why the analysis of stochastic layers is very instrumental in identifying the properties of the system. It is easy to imagine that in cases more complicated than that of pendulum stochastic, layers form an intricate net of "channels" in the phase space (Fig. 5). This net may be called a stochastic web. It has a finite width. If the initial condition of a particle is such that it is inside one of the web channels, then its further motion is random walkings along web channels. On the contrary, if a particle was within a web cell its trajectory will be regular and its Poincaré mapping will produce an invariant curve inscribed in the web cell. In 1964 Arnold predicted the existence of a stochastic web as a universal property

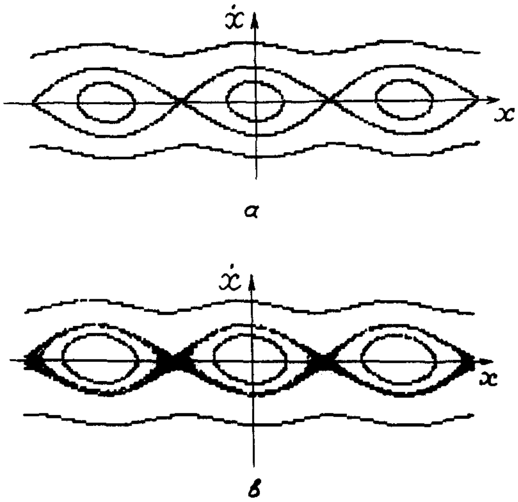

Fig. 4. Phase plane of a pendulum (a) and stochastic layer formation in a vicinity of separatrix (b).

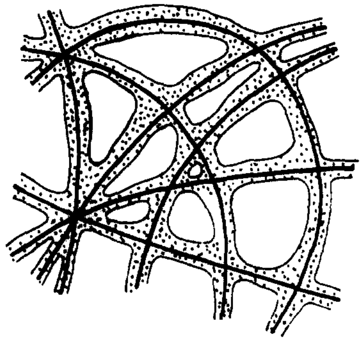

Fig. 5. Separatrix network (bold lines) and stochastic web (the dotted region).

of dynamic systems with more than two degrees of freedom (systems with N degrees of freedom are characterized by N coordinates and N impulses) [12]. It is now evident that a stochastic web, infinite in size, may arise even in the case when a system with only one degree of freedom is perturbed by a periodic force [8, 10]. In particular, it is exactly this case that the already mentioned problem of a charged particle, moving in a constant magnetic field and in the field of an electrostatic wave packet propagating perpendicularly to the magnetic field, can be reduced. In that problem the existence of a qth order resonance is the condition for the web to appear.

The appearance of an infinite stochastic web has several implications. Some of them are obvious. The web may exist at a minimal possible dimensionality of the system as, for $N = 1$, all systems have regular trajectories. That the web is infinite means a possibility of a diffusive acceleration of particles along the web channels up to very high velocities. However, the most unusual properties of the web are associated with its geometry and its symmetry.

SECTION 4

The Poincaré mapping for a particle in the magnetic field and the field of a perpendicular wave packet, for certain conditions, is presented by:

$$\hat{M}_\alpha : \begin{cases} u_{n+1} = (u_n + K \sin v_n)\cos\alpha + v_n \sin\alpha, \\ v_{n+1} = -(u_n + K \sin v_n)\sin\alpha + v_n \cos\alpha, \end{cases} \tag{2}$$

where (v, u) are the coordinates of the vector $\mathbf{R}$ on the phase plane; K is the parameter proportional to the wave packet amplitude; α is the angle by which a particle turns in the magnetic field between two consecutive kicks caused by the wave packet. The mapping $\hat{M}_\alpha$ relates the coordinate v_n and the particle impulse u_n at the time instant t_n, a kick acts upon the particle, with the coordinate v_{n+1} and the impulse u_{n+1} at the time t_{n+1} of the next kick acting. If all the time intervals between kicks are assumed to be unity, then

$$t_n = n.$$

The Hamiltonian of the particle motion has the form

$$H = \tfrac{1}{2}(\dot{x}^2 + \alpha^2 x^2) - \alpha K \cos x \sum_{n=-\infty}^{+\infty} \delta(t-n) \tag{3}$$

and the equation of motion for the particle is

$$\ddot{x} + \alpha^2 x = \alpha K \sin x \sum_{n=-\infty}^{+\infty} \delta(t-n). \tag{4}$$

It is evident from equations (3) and (4) that they describe the motion of a linear oscillator with the frequency α when subjected to a perturbation in the form of periodical δ-pulses. The same equations describe the motion of a particle in the magnetic field with the Larmor frequency α while the perturbation is a wave packet with an infinite number of harmonics, which are propagating along the x-axis. Comparing the solutions after two successive kicks permits the differential equation of motion, equation (4), to be replaced with a different equation. This gives mapping (2), where $v = x$, $u = \dot{x}/\alpha$.

We deliberately wrote equation (3) for the Hamiltonian H which gives origin to the mapping $\hat{M}_\alpha$. It is obvious from the form of H that its first term (in parentheses) describes only the rotation of the particle. Trajectories of rotation on the phase plane are characterized by a degenerate rotation symmetry. The second term of the Hamiltonian describing the kicks is invariant to the coordinate shift $x \to x + 2\pi m$ (m is the integer). In the absence of the magnetic field particle trajectories in the phase plane would have a translation symmetry like the phase portrait of the pendulum in Fig. 4. Thus, the Hamiltonian H realizes the rotation and translation symmetries interaction. The parameter K characterizes the intensity of this interaction.

For small K the interaction between rotational and translational symmetries is insignificant, and we face the question: what should the phase portrait of a dynamic system look like to implement both symmetries together? Though the parameter K is small, the interaction of symmetries should be most effective if the resonance condition is valid

$$\alpha = 2\pi/q, \tag{5}$$

where q is the integer.†

The above considerations show that we have achieved some success, though no specific results have been obtained yet. By this, we mean that we managed to formulate the problem of symmetries interaction as a problem of determining the phase portrait of a certain dynamic system. In other words, the problem of such tiling of the plane with weakly interacting rotation and translation symmetry is solved by determining invariant sets of mapping (2) over the phase plane (u, v). It only remains to consider these sets.

SECTION 5

If the value q belongs to the set

$$\{q_c\}: q = 2, 3, 4, 6,$$

then and only then the rotation and the translation symmetries may coexist on the plane. The mapping (2) at $q \in \{q_c\}$ generates simple square and hexagonal grids on the plane. Figure 6 shows such grids (for $q = 3$ and $q = 6$ the same structure results which is called a "kagome lattice").

For small values of parameter K in $\hat{M}_\alpha$ and for the resonance values of $\alpha = 2\pi/q$ there are stochastic webs with a crystal structure on the phase plane. The web is not thick but finite. It is generated by a single trajectory. If the initial coordinate of a particle (u_0, v_0) is chosen so that it is lying within the area occupied by the web, then iterations of this point according to mapping (2) will create these pictures illustrated in Fig. 6. If the initial condition (u_0, v_0) is somewhere inside the cells, formed by the web, then on the phase plane there will appear within the cells closed orbits which are cross-sections of invariant tori.

SECTION 6

The mapping $\hat{M}_q \equiv \hat{M}_{\alpha = 2\pi/q}$ has many striking properties. One of them is that the stochastic web exists at any $q \neq 2$ and, obviously, at arbitrarily small K [8, 9]. This means, in particular, that at $q \notin \{q_c\}$ new kinds of structure should originate. That they appear at resonance values of α [see equation (5)] is the most nontrivial fact in the theory, and even at a first glance (Fig. 7 for $q = 5$

†Cases with $\alpha = 2\pi p/q$ (p, q are integers) somewhat change the dynamics of particles, but do not change the symmetry of the phase portrait.

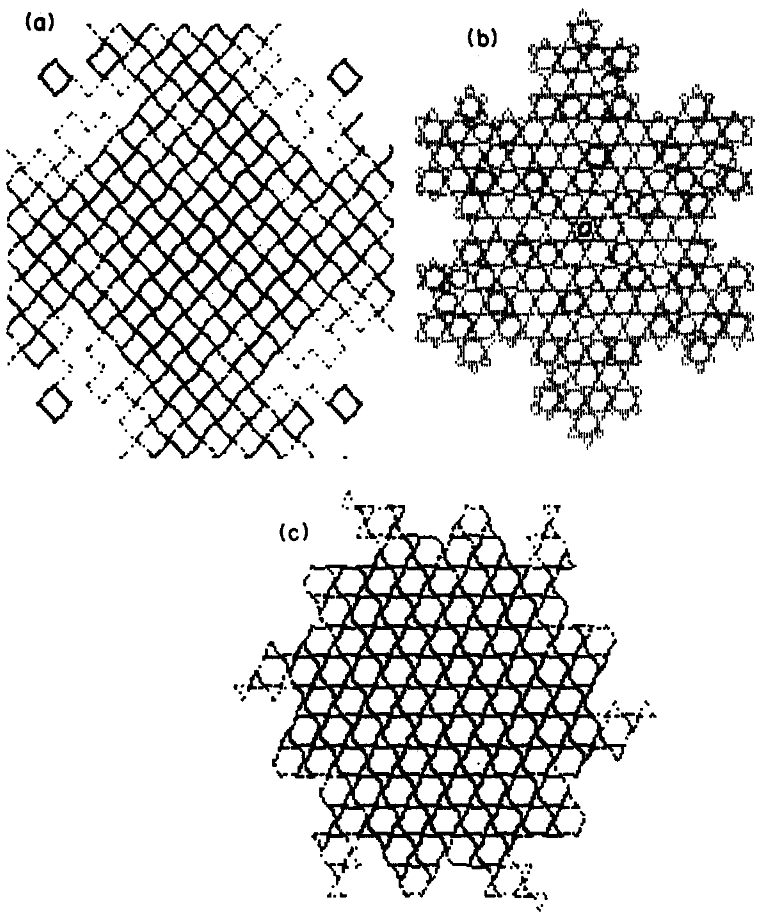

Fig. 6. Square and hexagonal lattices. (a) $q = 4$; (b) $q = 3$; (c) $q = 6$.

and $q = 7$) it seems that we meet with a new type of symmetry.† We will refer to it as a q-fold symmetry. Patterns in Fig. 7 as well as in Fig. 6 are drawn again by a single particle trajectory and this, also amazing, fact implies that, with simple electromagnetic field configurations, we may construct patterns the complexity of which could not have even been imagined until recently. Furthermore, if we now inspect Figs 6(b), 7(a) and 7(c) it becomes obvious that the "snowflakes" imaged there are typical fractals. Hence, the trajectories of a particle, generated by the mapping $\hat{M}_q$, may look like such monster-curves which were earlier assigned to the category of "formal curves".

The webs in Fig. 7 have approximately a rotative symmetry. On the one hand, their approximate character is related with the finite time over which the image is obtained. Owing to this, not all of its details have been obtained during that time. On the other hand, the web thickness is finite. Its width irregularly varies over the plane, and it also slightly perturbs the symmetry. However, the most essential question is which structure, regular or amorphous, originates at $q \notin \{q_c\}$. The answer is obvious after the Fourier spectrum of the web is derived, that is, when its X-ray pattern is obtained. According to Fig. 8, we deal with an ordered structure. The spectral pattern in Fig. 8(c) is very similar to those recorded for real quasicrystals [3]. This is a good reason for us to believe that the symmetry of the web corresponds to a quasicrystal symmetry.

SECTION 7

A more rigorous derivation of this conclusion is possible [9, 13]. We have already mentioned that a stochastic web forms after the separatrix net is destroyed. It may also be said that the separatrix

†The "window" in Fig. 7(c) remains open for a long time (for about 10^6 iteration steps).

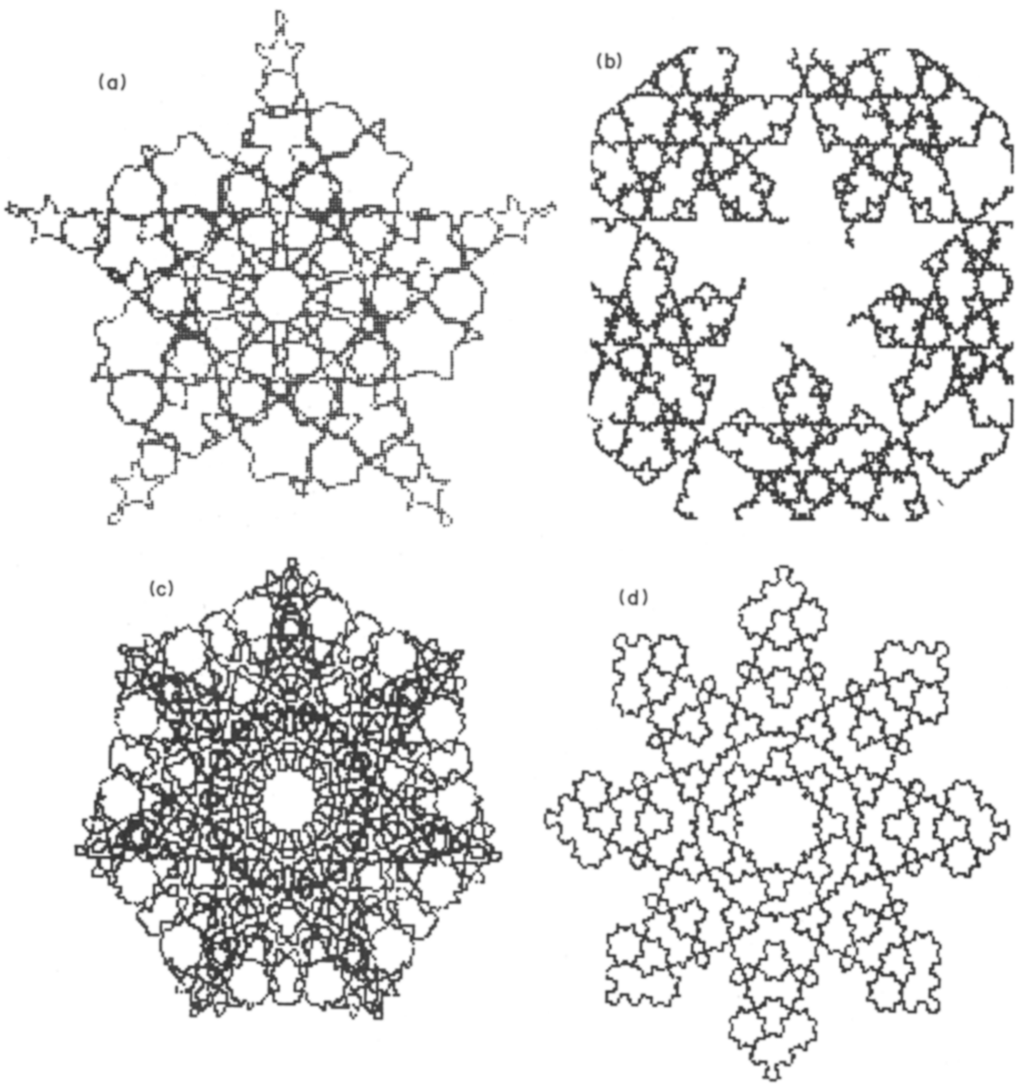

Fig. 7. Stochastic web with 5-fold symmetry (a, b), 7-fold symmetry (c) and 8-fold symmetry (d). The cases (a) and (b) differ in initial conditions.

net constitutes a web skeleton. Thus, we may try to outline this skeleton which should be a certain, infinitely thin frame of the web from which minor features are removed. We may go further still and outline the original dynamic system for which the web skeleton forms a separatrix net. Note that we are consistent when we try to get the skeleton not as a geometric object but rather as an invariant set on the phase plane for a certain appropriate dynamic system.

This program was employed in Refs [9, 13]. Replacing the variables and averaging equation (3) yield a "seed Hamiltonian"

$$H_q = \sum_{j=0}^{q} \cos(\mathbf{R}\mathbf{e}_j), \tag{6}$$

where $\mathbf{R} = (v, u)$ and the "hedgehog" $\{\mathbf{e}_j\}$ consists of unit vectors which form a regular q-star

$$\mathbf{e}_j = \left(\cos 2\pi \frac{j}{q}, \sin 2\pi \frac{j}{q}\right). \tag{7}$$

Equation (6) coincides with the one used for the free energy of quasicrystals from phenomenological considerations [14–16]. For instance, for $q = 4$:

$$H_4 = 2(\cos u + \cos v).$$

This Hamiltonian has a square separatrix net. It is often employed in the physics of solids to describe the motion of electrons in crystals. When $q = 3$ or $q = 6$:

$$H_3 = \frac{1}{2} H_6 = \cos u + \cos\left(\frac{1}{2} u + \frac{\sqrt{3}}{2} v\right) + \cos\left(\frac{1}{2} u - \frac{\sqrt{3}}{2} v\right).$$

This Hamiltonian's separatrix net is a kagome lattice. It coincides with the stream function for the thermal Rayleigh–Bénard convection. In this case, hexagonal convective Bénard cells are formed in a fluid.

Thus, we have obtained for $q \in \{q_c\}$ just what we needed: a separatrix net of "seed Hamiltonian" has an ideal 3-, 4- or 6-fold symmetry. However, in the case of quasisymmetry the situation is not that easy.

SECTION 8

One of the main structural elements of H_q are singular points of saddle type. On the phase plane they are joined by specific trajectories, separatrices. Not very often could separatrices form a single net, that is, a web. For $q \in \{q_c\}$ such a net forms, this happens since the same $H_q = E_c$ ($E_c = 0$ for $q = 4$ and $E_c = -1$ for $q = 3$) are assigned to all saddles. In other words, in the case of crystal symmetry all saddles are in the plane of one level $H_q = \text{constant} = E_c$. That is why they may be connected by a single separatrix net.

If $q \notin \{q_c\}$ the saddles are at different levels of H_q [13]. Owing to this fact, separatrix net common for the entire plane cannot be built in the plane of one particular level of H_q-values. It is quite obvious from Fig. 9, where better resolving power of the picture shows that many intersections within the net on the plane do not exist really. This is very essential for our understanding the origin of the web with quasisymmetry. The Hamiltonian at $q \in \{q_c\}$ has a skeleton with crystal symmetries $\{q_c\}$, but does not have skeletons with quasisymmetries at $q \notin \{q_c\}$. However, even very small perturbation δH_q blurs small gaps between individual loops in separatrices, and a single unlimited web appears, via which propagation is possible over any arbitrarily long distance.

With this explanation it is relatively easy to obtain a quasiskeleton from H_q in the case of quasisymmetry [13]. To do this, that value E_0 should be determined, to which the peak of the distribution of the number of saddles as a function of $H_q = E$ values corresponds. Next, a set of points should be obtained which belong to the range of levels $E \in (E_0 - \Delta E, E_0 + \Delta E)$. This is equivalent to a weak blurring of separatrices lying at the level E_0. It is the thus derived picture which is a quasiskeleton (Fig. 10).

SECTION 9

The new type of tiling the plane with quasicrystal symmetry, or quasisymmetry, is fully determined by the tiling generator $\hat{M}_q$:

$$\hat{M}_q: \begin{cases} \bar{u} = (u + K \sin v)\cos \dfrac{2\pi}{q} + v \sin \dfrac{2\pi}{q}, \\ \bar{v} = -(u + K \sin v)\sin \dfrac{2\pi}{q} + v \cos \dfrac{2\pi}{q}. \end{cases} \tag{8}$$

The smaller the K values, the more regular the patterns are.

The symmetry of a tiling may be controlled by varying only one parameter q. This is why we are now able to show what an arbitrary q-sided snowflake can look like, whereas Figs 7(a) and 7(c) given here demonstrate actual possible shapes of a pentagonal snowflake (the problem that was already puzzling Keppler) and of a heptagonal snowflake. The plane cannot be paved with regular q-gons if $q \notin \{q_c\}$. However, it can be paved so that basic paving elements be almost regular q-gons (or $2q$-gons), q-gonal stars and something else. That "something else" includes other figures; however, a stochastic web is the most important element in the paving process. It is this web that helps to remove various inconsistencies during the paving, though it can be made

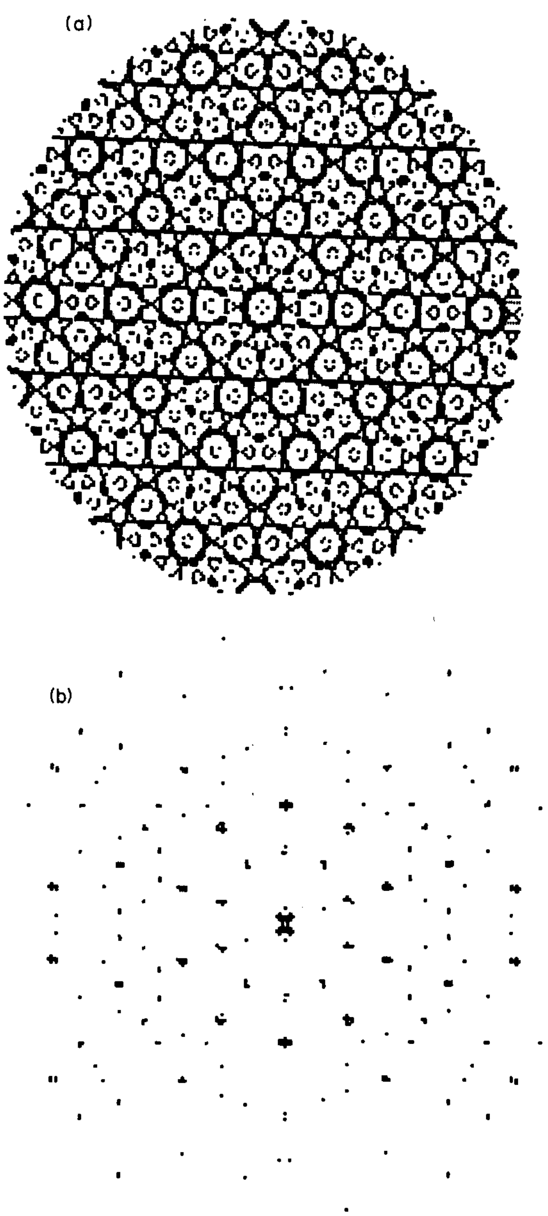

Fig. 8. Web skeleton for $q = 5$ (a) and its Fourier spectrum (b).

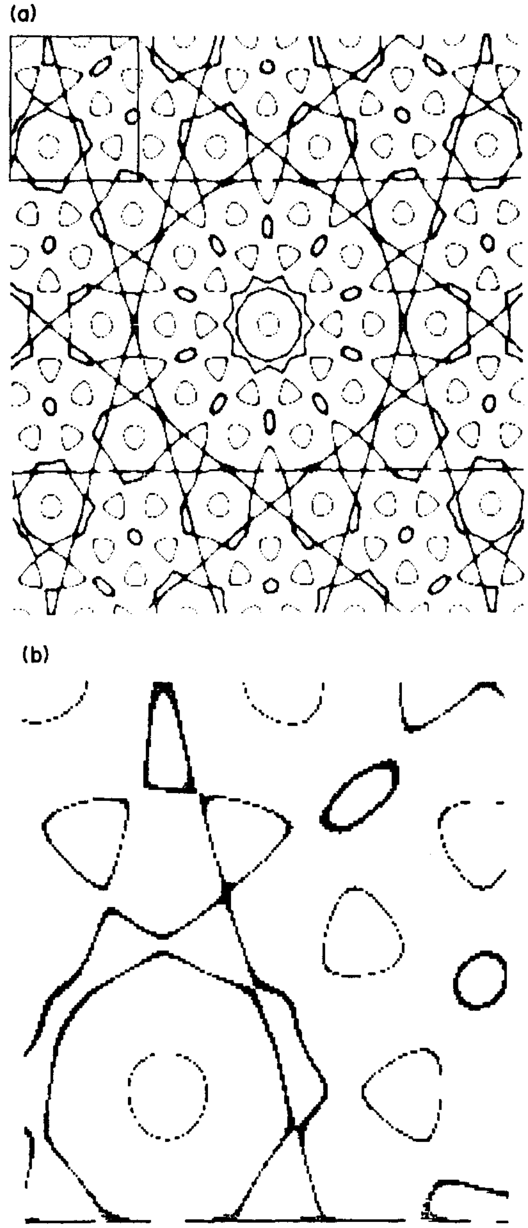

Fig. 9. Web skeleton for $q = 5$ (a) and its part in the square magnification (b).

arbitrarily thin. Of course, the paving method depends on how thick the web is. The finitely thick web smoothes individual inconsistencies of tilings with quasisymmetries, and the fact that the web is fractal makes us approach quite different possibilities of symmetry analysis. Whatever idea we may now have about a quasicrystal it is unlikely that different symmetries can coexist in this crystal without their interaction. That is why, for instance, that the Penrose tiling of the 5-fold symmetry, which covers the plane only with two different rhombs, may be good approximation to a real structure whose roughness, indeed, leads to fractality. It is for this reason that in such cases the quasisymmetry is preferable to the exact symmetry.

We have seen already that the mapping $\tilde{M}_q$ can be less accurate coarse-grained by remvoing some small details. Then there appears in the phase plane a pattern with a rotation symmetry, q-fold for even q and $2q$-fold for odd q (Figs 9 and 10). These patterns are obtained by the same principle as used for mountain relief on maps when regions in the same height range are shown by the same

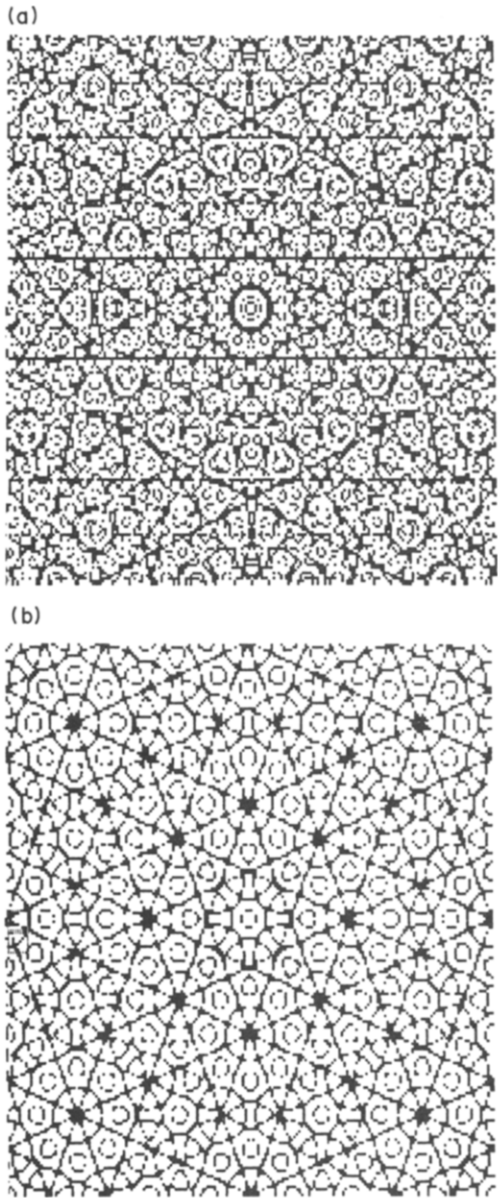

Fig. 10. Quasiskeletons for $q = 7$ (a) and $q = 8$ (b).

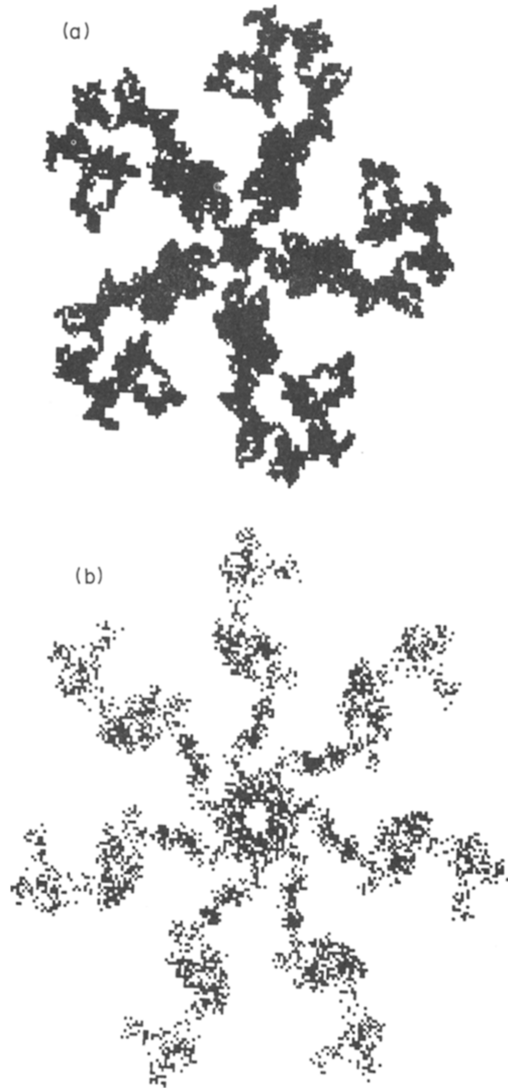

Fig. 11. Fractal trees for $q = 5$ (a) and $q = 7$ (b), result from the particle diffusion.

color. In the former case the energy of a particle is used instead of height. Thus, derived reliefs have very distinct cluster regions in the form of q- or $2q$-gons. What is more, these clusters are formed by a set of almost straight lines, parallel and turned q times around the center. Fairly essential here is the thickness of lines. If allowed to approach zero, many elements in the structure will disappear. Hence an important aspect of quasisymmetry is its nonideality. Not only does this property spoil the symmetry, it simultaneously leads to quasisymmetry in the mapping. The rotation symmetry in those patterns and the presence of a set of straight lines point to the existence of the long-order though the translational symmetry lacks. On the other hand, aperiodic character of star clusters is similar to that of fluids. The higher the q, the closer the quasicrystal structure is, in some of its properties, to fluids [13].

If the parameter K is made larger in $\hat{M}_q$ the interaction of the two symmetries will become stronger and the stochastic web becomes wider. With its further growth, small cells of the web will be overgrown with the areas of chaotic dynamics and some symmetry properties of the web will be destroyed. Intense Brownian motion of particles beings (in case there are no random forces!)

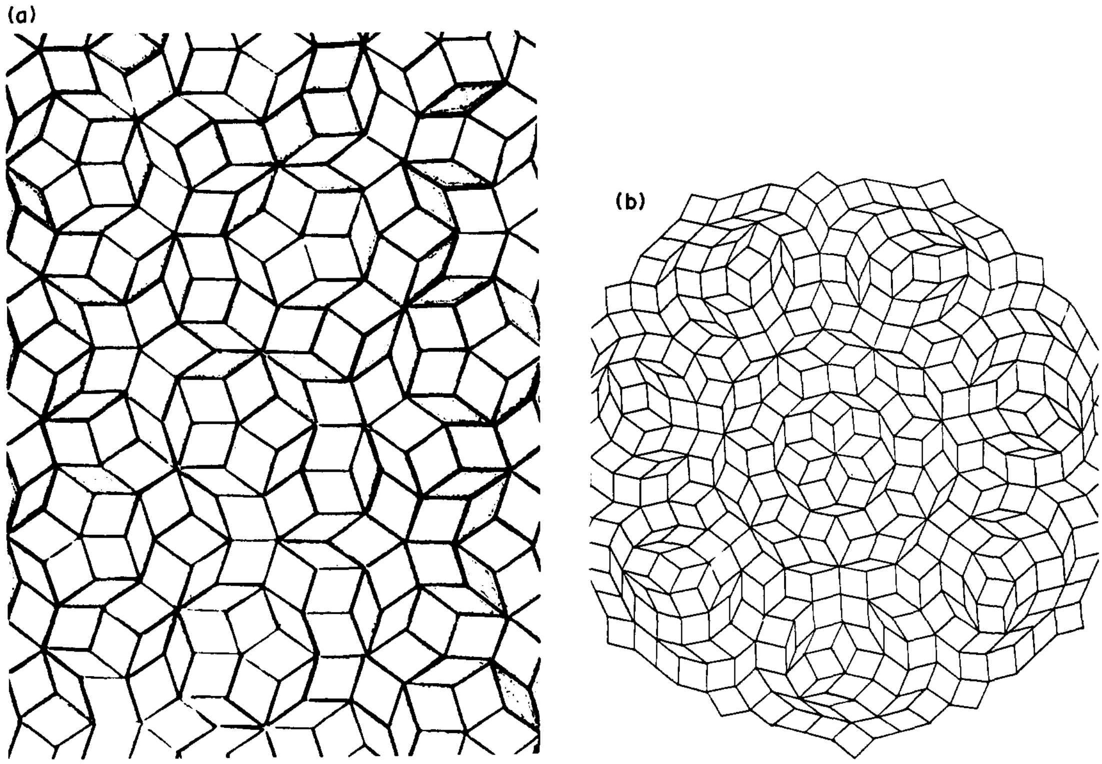

Fig. 12. Pentagonal tiling (Penrose tiling) (a) and heptagonal tiling (b).

which is imaged in the phase plane (by the trajectory puncturing the plane) as a growing fractal tree. The tree preserves a rotation q-fold symmetry and may be regarded as a result of particle diffusion process (Fig. 11).

Another extreme in simplifying the structures with quasisymmetry is building of parquets of the finite number of rhombs. Examples of pentagonal structure (Penrose tilings according to MacKay [6]) and heptagonal structure [13] are given in Fig. 12. At $q = 5$ the tilings are made with two rhombs, one angle of which is $\pi/5$ or $2\pi/5$. At $q = 7$ three rhombs are used for the tiling whose minor angles are $\pi/7$, $2\pi/7$, $3\pi/7$, respectively. This rule may obviously be extended in the similar manner for arbitrary q.

Tilings in Fig. 12 are quite definitely related to reliefs for $q = 5$ [Fig. 9(a)] and $q = 7$ [Fig. 10(a)]. There exist algorithms of such decoration of reliefs which turn them into parquets (for the case $q = 5$, see Ref. [9]).

SECTION 10

We need one more step which will reveal one of the most fundamental properties of quasisymmetry. The existence of symmetries is associated with a certain procedure for idealizing real processes. To get a quasisymmetry, the level of idealization should not be too high, nor should this idealization be too trivial. Up to now we followed the same sequence: web–relief–parquet. Structural metamorphoses, in the sequence, made the overall picture less detailed, thus, helping to single out very accurately some of its properties.

Let us give one more result of such coarse-graining. It is the determination of X-ray pattern of structural relief of the type illustrated by Fig. 10.

To do this a two-dimensional Fourier transform of a respective relief should be obtained. We now consider, for instance, an area in the central part of the relief at $q = 11$ and its Fourier transform (Fig. 13). Not only does the spectral image show that it is an ordered structure but also that it has the 11-fold rotation symmetry.

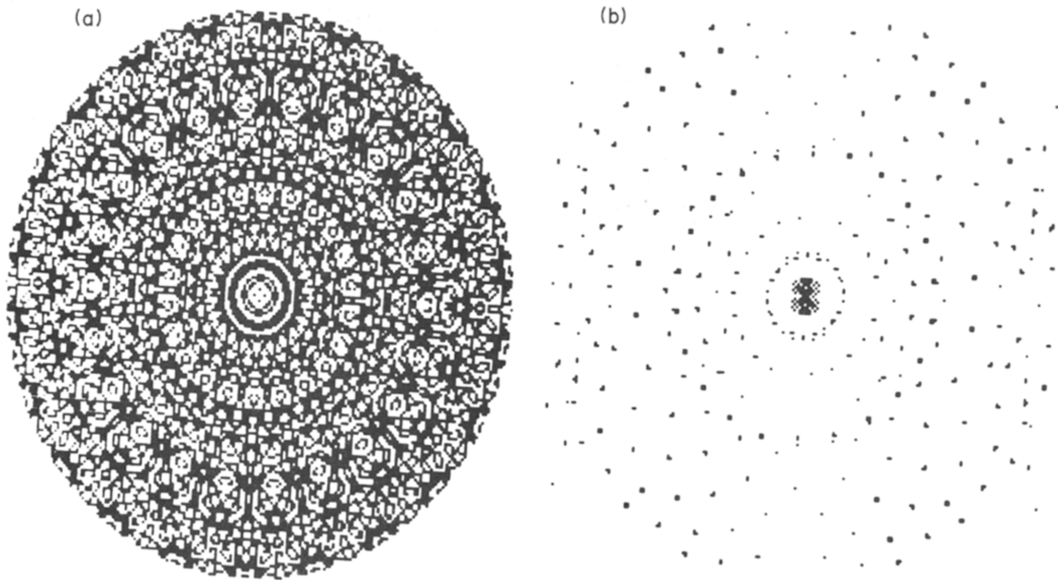

Fig. 13. Central part of the skeleton for $q = 11$ (a) and its Fourier spectrum (b).

We now select an area of relief of the same size and scale in a different part of the plane, far enough from the central part in Fig. 13(a). Figure 14(a) is the example. From what this area looks like we can say nothing about its symmetry, nor about the level of order of the entire relief whose part it is. What is more, it is natural to refer to this pattern as to a texture, rather than a crystal. However, the appropriate Fourier analysis [Fig. 14(b)] gives an amazing result. Shown there is the spectrum of the ordered structure with an 11-fold rotation symmetry which almost completely coincides with the Fourier spectrum of the area in the central part in Fig. 13(b).

This example shows that sufficiently large elements of a quasisymmetric tiling have very similar coarse-grained Fourier spectra. Coarse-graining involves cutting off the harmonics with very small amplitudes and smoothing other harmonics over amplitudes.

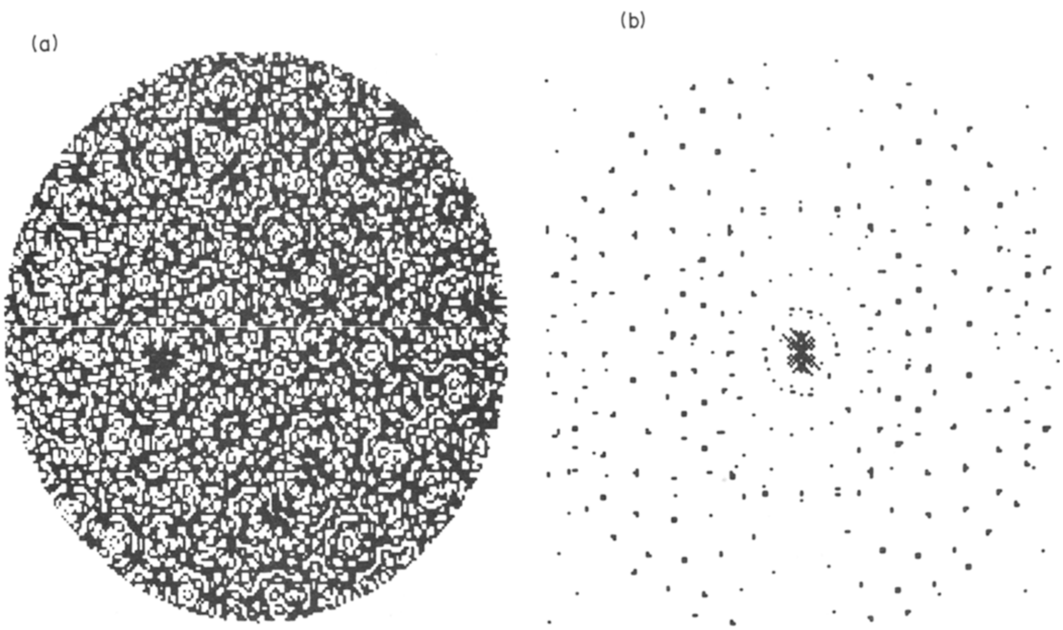

Fig. 14. Noncentral part of the skeleton for $q = 11$ (a) and its Fourier spectrum (b).

SECTION 11

The problem of charged-particle dynamics in the magnetic field and in the field of the plane wave packet is very popular in the theory of charged-particle accelerator, in plasma where oscillations are excited and in some problems of hydrodynamics. This problem seems very specific, nevertheless, a new class of tiling generators with q-fold rotation symmetry developed within it, which is of universal nature. This class is related with the structure of quasicrystals, besides, there is one more important application of tiling generators: these are possible types of pattern in gases and fluids, which are formed at the transition from a laminar (regular) to a chaotic (turbulent) state.

Random walkings of a particle on the plane generate a certain pattern. It grows as snowflakes do or, in a more general case, its growth reminds that of crystals or quasicrystals. Similar problems are also met in the finite automata theory. Hence the tiling generator $\hat{M}_q$ may be an unusually convenient tool for drawing various ornaments. Thus, the problem of the intricate dynamics of a charged particle opens up one more vista, i.e. computer graphics. With a color display all kinds of computer-aided configurations and ornaments can be plotted, if only a specific algorithm can be given which is a pattern generator. Search for most interesting algorithms has become a kind of sport, with its own program-favorites and its own business aspects. The new generator $\hat{M}_q$ may generate a great number of various ornaments with any order of symmetry. This number drastically increases because the patterns that thus appear are fractal. When the scale of the image is changed the typical features of an elementary information-carrying cell on display, that is, of a pixel, also change. Owing to this, the latter may be differently colored. Because of fractality the change in the resolving power of the instrument is accompanied by the change in the structure of the object

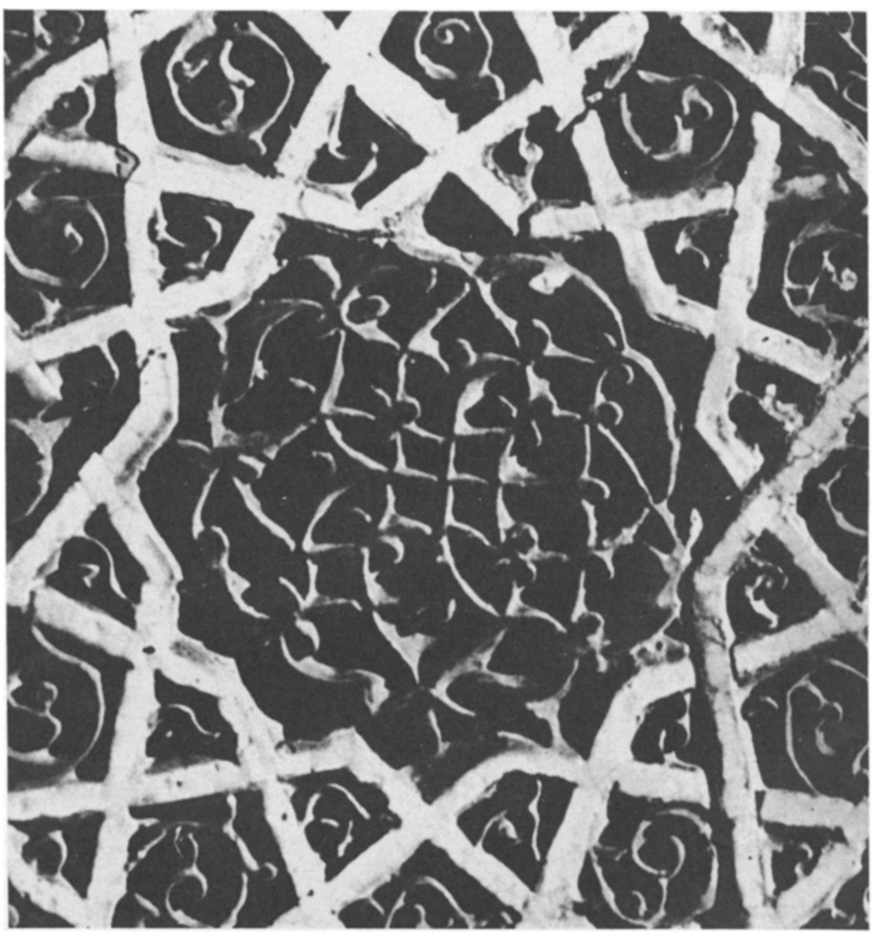

Fig. 15. Typical element of Muslim ornaments.

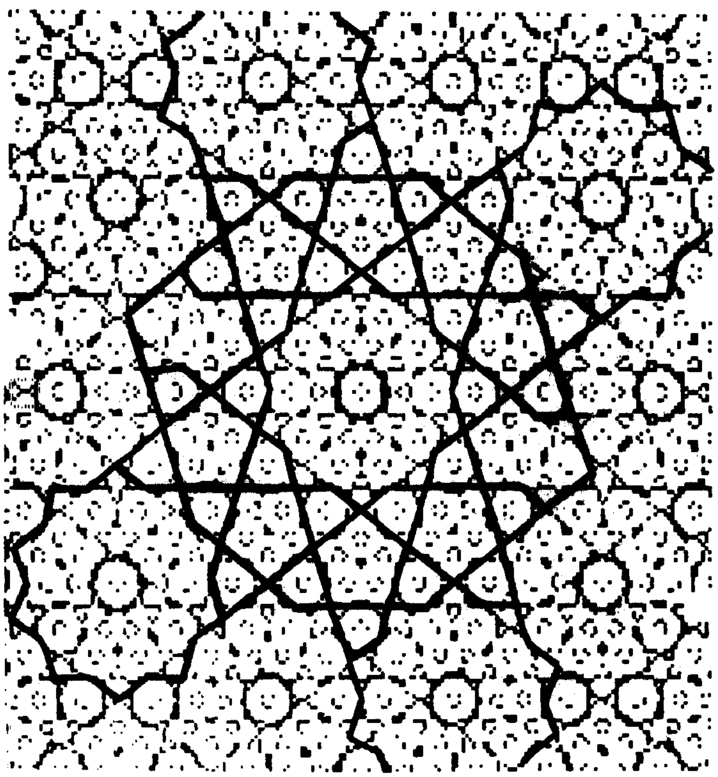

Fig. 16. "Decoding" of the ornament from Fig. 15 by the 5-fold symmetry skeleton.

it determines. We again come back to the problem of ornaments, though now this variety is described by real physical processes and is implemented in such real objects as quasicrystals, foam, hydrodynamic pattern. In this respect, not only does a new computer graphics produced with mapping $\hat{M}_q$ have a certain aesthetic configuration, but it also reflects the beauty of the real physical world.

Now that we know certain typical features of the tiling generator $\hat{M}_q$, we can approach quite differently the secrets of the art of the Moslem painters of the past who created a tremendous number of various ornaments which make us follow with attention and amazement the extraordinary intricacy and sophistication with which a craftsman drew the fundamental lines in his pattern. If the secrets of symmetric tiling are known it is easy to perceive how a given ornament was drawn. Figure 15 is an example of an element of a fairly typical ornament. It has a local-decagon. The scheme of the ornament is easily obtainable if a relief with the same symmetry and derived from $\hat{M}_q$ ($q = 5$) is applied. The respective decoration procedure is shown in Fig. 16. The painter, however, did not follow the scheme and the 5-fold symmetry was destroyed. The entire ornament (Fig. 17) is like a square lattice. One can find an element of nonperiodic pentagonal ornament in one of the frescoes of the ancient Moorish palace, Alhambra, in Granada.

Examples of the generation of ornaments determined by the symmetry of a "quasicrystal" type or, to be more exact, having certain quasisymmetry, open up new ways for analysis of geometrical properties of nature. One of the striking features of the new forms of symmetry is that, with the help of disorder elements, they produced a long-range order in structures. These disorder elements are the channels of the stochastic web, which smooth over minor disagreements between structure's bricks. Thus, an almost dense packing is achieved with only certain minimal gaps. Dynamic models help to separate the areas of elements packed and the areas of gaps in between according to the type of motion within them. This became possible with the help of two radically opposite types of motion, i.e. regular and random. It is just the essence of a new view on the long-range order organization and it may be due to the presence of chaotic disorder elements. Real physical processes, which include weak interaction of two mutually excluding symmetries, give rise to a weak

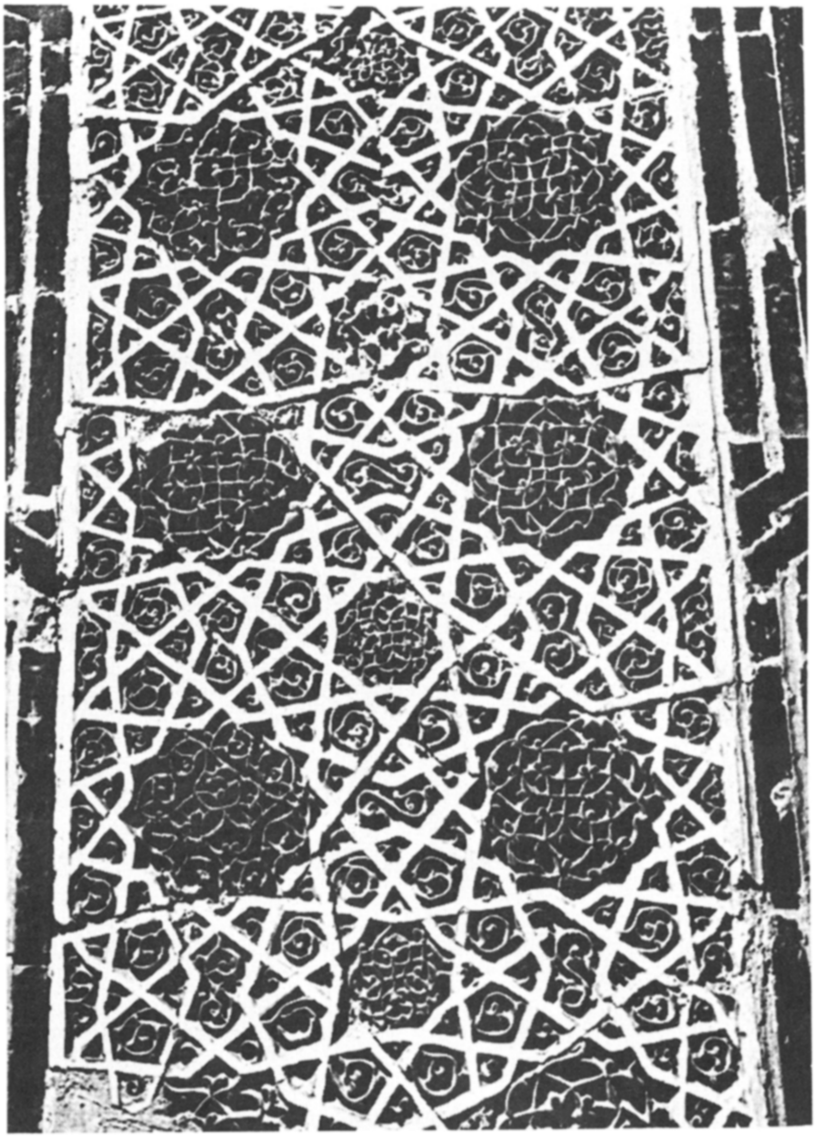

Fig. 17. Periodical tiling which includes the element from Fig. 15.

stochasticity. It is the tribute that should be paid for the maintaining of order in the form of quasisymmetry!

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